Some Fixed Point Results through Semi Compatibility & Weakly Compatibility in 2-Banach Space

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Abstract:

. *This paper deals with few common fixed point results through semi compatibility in complete 2-Banach space which generalize the results of Som (2005), Amalendu Choudhury and T. Som (2011) and Mukti Gangopadhyay, Mantu Saha & A.P. Baisnab (2009).*

Key Words:

Compatible maps, Semi-compatible maps, weak compatible maps, 2-Banach Space, Convergent sequence, Cauchy sequence and fixed point.

1. Introduction:

Theory of 2-Banach spaces was investigated by S. Gahler and K. Iseki who had proved some fixed point theorems in 2-Banach spaces. Y.J. Cho, N. Huang and X. Long proved some fixed point theorems for nonlinear mappings in 2-Banach spaces. M.S. Khan and M.D. Khan worked for Involutions with fixed points in 2-Banach spaces. In this concept some basic fixed point results are given by Gahler in 1960. In this paper we achieved some results in fixed point theory in a 2-Banach space by working with semi compatibility in a complete 2-Banach space.

2. Preliminaries:

Definition 2.1 Let X is a real linear space and $\|.,.\|$ be a non-negative real valued function defined on X satisfy the following conditions:

(i) $\|x,y\| = 0$ If and only if x and y are linearly dependent.
(ii) $\|x,y\| = \|y,x\|$ for all $x, y \in X$.
(iii) $\|x,ay\| = |a|\|x,y\|$, a being real and for all $x, y \in X$.
(iv) $\|x,y+z\| \leq \|x,y\| + \|y,z\|$ for all $x, y, z \in X$.

Then $\|.,.\|$ is called a 2-norm and the pair (X, $\|.,.\|$) is called a linear 2-normed space.

So a 2-norm $\|x,y\|$ always satisfies $\|x,y+ax\| = \|x,y\|$, for all $x, y \in X$ and all scalars a.

Definition 2.2

A sequence $\{x_n\}$ in a 2-normed linear space (X, $\|.,.\|$) is said to be convergent to an element $x \in$ X if $\|x_n - x, a\| = 0$ *as* $n \to \infty$ *and for all* $a \in X$.

Definition 2.3

If the sequence $\{x_n\}$ converges to x then $\|x_n - a, b\| = \|x - a, b\|$ as $n \to \infty$ *and for all* $a, b \in X$.

Definition 2.4

A sequence $\{x_n\}$ in a 2-normed linear space (X, $\|.,.\|$) is a Cauchy sequence if $\|x_m - x_n, a\| = 0$ as m, n$\to \infty$ and for all $a \in X$.

Definition 2.5

If a sequence $\{x_n\}$ is convergent sequence in a 2-normed linear space, then it is a Cauchy sequence.

Definition 2.6

A 2-normed linear space (X, $\|.,.\|$) is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.7

Two maps T and S are said to be commuting if $TSx = STx$ for all $x \in X$.

Definition 2.8

Let T and S be two self maps on a set X, if $Tx = Sx$ for some $x \in X$ then x is called coincidence point of T and S.

Definition 2.9

Let X is a 2-Banach space. T and S are said to be weakly compatible if they commute at their coincidence points. i.e., $Tx = Sx$ for some $x \in X \Rightarrow TSx = STx$.

Definition 2.10

Two self mappings T and S of a 2-Banach space X are called compatible if $\lim_{n\to\infty} \|STx_n - TSx_n, a\| = 0$ for all $a \in X$, when $\{x_n\}$ is a sequence in X such that $\lim_n Sx_n = \lim_n Tx_n = x$ for some $x \in X$.

Definition 2.11

Two self mappings T, S: X$\to X$ are said to be semi-compatible if $\lim_n \|TSx_n - S_x, a\| = 0$ *for* all $a \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n Sx_n = \lim_n Tx_n = x$ for some $x \in X$.

Definition 2.12

Let $\lozenge: R^+ \times R^+ \to R^+$ be a binary operation satisfying the following conditions:

1. $\lozenge$ is associative and commutative
2. $\lozenge$ is continuous.

Definition 2.13

The binary operation $\lozenge$ is said to be satisfy α-property if there exists a positive real number α such that a $\lozenge$ b $\leq \alpha \max\{a, b\}$ for all $a, b \in R^+$.

Propositions

3.1: If a sequence $\{x_n\}$ in a 2-normed linear space converges to *x* then every subsequence of $\{x_n\}$ also converges to the same limit *x*.

3.2: Limit of a sequence in a 2-normed linear space if it exists is unique.

3.3: Semi-compatibility of the pair (A, B) does not imply semi-compatibility of the pair (B, A). Hence, weak compatibility need not imply the semi-compatibility.

3.4: Compatibility does not imply semi-compatibility. A Class of Implicit Relation

Let F_5 be the class of upper semi-continuous functions on the right from $(R^+)^5 \to R$, such that for some $h \in (0,1)$

(i) $F(u, v, u, u, v) \geq 0 \Rightarrow v \leq hu$
(ii) $F(u, v, v, u, v) \geq 0 \Rightarrow v \leq hu$
(iii) $F(u, 0, u, u, 0) \geq 0 \Rightarrow u = 0$.

Theorems 4.1:

Let A, B, S and T is self maps of 2-Banach space (X, $\|.,.\|$) satisfying the following:

(1.1) $A(X) \subseteq T(X), B(X) \subseteq S(X)$
(1.2) There exists $h \in (0,1)$ such that $\|Ax - By, a\| \leq h[max\{\|Sx - Ax, a\|, \|Ty - By, a\|, \|Sx - Ty, a\|, \|Ty - Ax, a\|, \|Sx - By, a\|\}]$ for all $x, y, a \in X$.
(1.3) The pair (A, S) is semi-compatible and the pair (B, T) is weak compatible.
(1.4) One of A(X), T(X), B(X) or S(X) is complete.
(1.5) There exists $F \in F5$ such that

$$F[\|Ax - By, a\|, \|Ax - Ty, a\|, \quad \|Sy - By, a\|, \|Ax - Sy, a\|, \|By - Ty, a\|] \geq 0$$

Then A, B, S and T have a unique common fixed point in X.

Proof: Let x_0 be any point in X, then by condition (1.1) there exists $x_1, x_2 \in X$ such that $Ax_0 = Ty_1 = y_0$ and $Bx_1 = Sx_2 = y_1$.

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}, y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}, n = 1,2,3, \ldots$ ------- (1.6)

Putting $x = x_{2n}, y = x_{2n+1}$ in (1.2), we get,

$\|Ax_{2n} - Bx_{2n+1}, a\| \leq h[max\{\|Sx_{2n} - Ax_{2n}, a\|, \|Tx_{2n+1} - Bx_{2n+1}, a\|, \|Sx_{2n} - Tx_{2n+1}, a\|, \|Tx_{2n+1} - Ax_{2n}, a\|, \|Sx_{2n} - Bx_{2n+1}, a\|\}]$

$\|y_{2n} - y_{2n+1}, a\| \leq h[max\{\|y_{2n-1} - y_{2n}, a\|, \|y_{2n} - y_{2n+1}, a\|, \|y_{2n-1} - y_{2n}, a\|, \|y_{2n} - y_{2n}, a\|, \|y_{2n-1} - y_{2n}, a\|\}]$

$\|y_{2n} - y_{2n+1}, a\| \leq h[max\{\|y_{2n-1} - y_{2n}, a\|, \|y_{2n} - y_{2n+1}, a\|\}]$

If $\|y_{2n} - y_{2n+1}, a\| \geq \|y_{2n-1} - y_{2n}, a\|$ then $\|y_{2n} - y_{2n+1}, a\| \leq h\|y_{2n} - y_{2n+1}, a\|$,

This is a contradiction.

Hence, $\|y_{2n} - y_{2n+1}, a\| \le h\|y_{2n-1} - y_{2n}, a\|$.

Similarly, $\|y_{2n+1} - y_{2n+2}, a\| \le h\|y_{2n} - y_{2n+1}, a\|$.

Therefore, for all n even or odd, $\|y_n - y_{n+1}, a\| \le h\|y_{n-1} - y_n, a\|$

$$\le h^2\|y_{n-2} - y_{n-1}, a\|$$

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$$\le h^n\|y_0 - y_1, a\|$$

Hence, $\{y_n\}$ is a Cauchy Sequence.

Case (i)

Suppose that S(X) is a complete subspace of X, then the sequence $y_{2n} = Sx_{2n+1}$ is a Cauchy sequence in S(X) and hence it has a limit z (say).

Consequently its subsequence also converges to z.

Now, $z \in S(X)$, so there exists $w \in X$ such that z = Sw.

Since, (A, S) are semi-compatible, $\lim_{n\to\infty} ASx_n = Sz$.

Step I. putting $x = Sx_{2n}$, $y = x_{n+1}$ in (1.5), we get

$$F\begin{bmatrix} \|ASx_n - Bx_{n+1}, a\|, \|ASx_n - Tx_{n+1}, a\|, \|Sx_n - Bx_{n+1}, a\|, \\ \|ASx_n - Sx_{n+1}, a\|, \|Bx_{n+1} - Tx_{n+1}, a\| \end{bmatrix} \ge 0$$

Letting $n \to \infty$, we get

$$F[\|Sz - z, a\|, \|Sz - z, a\|, \|z - z, a\|, \|Sz - z, a\|, \|z - z, a\|] \ge 0$$

That is, $F[\|Sz - z, a\|, \|Sz - z, a\|, 0, \|Sz - z, a\|, 0] \ge 0$.

$\Rightarrow \quad Sz = z.$

Step II. Putting $x = z, y = x_{n+1}$ in (1.2), we get

$$\|Az - Bx_{n+1}, a\| \le h\left[max\left\{\begin{matrix} \|Sz - Az, a\|, \|Tx_{n+1} - Bx_{n+1}, a\|, \\ \|Sz - Tx_{n+1}, a\|, \|Tx_{n+1} - Az, a\|, \|Sz - Bx_{n+1}, a\| \end{matrix}\right\}\right]$$

Taking as $n \to \infty$, $\|Az - z, a\| \le h[max\{\|z - Az, a\|, 0, 0, \|z - Az, a\|, 0\}]$

$\Rightarrow \quad Az = z = Sz.$

Step III. As $A(X) \subseteq T(X)$, there exists some $u \in X$, such that $z = Az = Tu$.

Putting x = x_{2n}, y = u in (1.2), we have

$$\|Ax_{2n} - Bu, a\| \le h\left[max\left\{\begin{matrix} \|Sx_{2n} - As_{2n}, a\|, \|Tu - Bu, a\|, \|Sx_{2n} - Tu, a\|, \\ \|Tu - Ax_{2n}, a\|, \|Sx_{2n} - Bu, a\| \end{matrix}\right\}\right]$$

As $n \to \infty$, we get

$$\|z - Bu, a\| \le h\left[max\begin{Bmatrix}\|z - z, a\|, \|z - Bu, a\|, \|z - z, a\|, \\ \|z - z, a\|, \|z - Bu, a\|\end{Bmatrix}\right]$$

$$= h[max\{0, \|z - Bu, a\|, 0, 0, \|z - Bu, a\|\}]$$

This implies $Bu = z = Tu$.

But (B, T) is weak compatible so BTu = TBu i.e., $Bz = Tz$.

Step IV. By putting $x = z, y = z$ in (1.2) and assuming that $Az \neq Bz$, we have

$$\|Az - Bz, a\| \le h\left[max\begin{Bmatrix}\|Sz - Az, a\|, \|Tz - Bz, a\|, \|Sz - Tz, a\|, \\ \|Tz - Az, a\|, \|Sz - Bz, a\|\end{Bmatrix}\right]$$

$$= h\left[max\begin{Bmatrix}\|z - z, a\|, \|Bz - Bz, a\|, \|Az - Bz, a\|, \\ \|Bz - Az, a\|, \|Az - Bz, a\|\end{Bmatrix}\right]$$

That is, $\|Az - Bz, a\| \le h\|Az - Bz, a\|$

This is a contradiction

Therefore, $Az = Bz = z$.

Combining all the results we get $z = Az = Sz = Bz = Tz$. Thus in this case z is a common fixed point of A, B, S and T.

Case (ii) . Let T(X) be complete. Then $y_{2n} = Tx_{2n+1}$ converges to some $z \in T(X)$. Hence there exists $w \in X$, such that $z = Tw$.

Step I. Putting $x = x_{2n}, y = w$ in (1.2), we get

$$\|Ax_{2n} - Bw, a\| \le h\left[max\begin{Bmatrix}\|Sx_{2n} - Ax_{2n}, a\|, \|Tw - Bw, a\|, \|Sx_{2n} - Tw, a\|, \\ \|Tw - Ax_{2n}, a\|, \|Sx_{2n} - Bw, a\|\end{Bmatrix}\right]$$

Letting $n \to \infty$, we get

$$\|z - Bw, a\| \le h\left[max\begin{Bmatrix}\|z - z, a\|, \|z - Bw, a\|, \|z - z, a\|, \\ \|z - z, a\|, \|z - Bw, a\|\end{Bmatrix}\right]$$

$\Rightarrow \quad Bw = z = Tw$.

But (B, T) is weak compatible, so $BTw = TBw$. That is $Bz = Tz$.

Step II. As (A, S) is semi compatible so $\lim_{n\to\infty} ASx_n = Sz$.

Putting $x = Sx_{2n}, y = z$ in (1.5), we get

$$F\begin{bmatrix}\|ASx_n - Bz, a\|, \|ASx_n - Tz, a\|, \|Sz - Bz, a\|, \\ \|ASx_n - Sz, a\|, \|Bz - Tz, a\|\end{bmatrix} \ge 0$$

Letting $n \to \infty$, we get

$$F[\|Sz - Bz, a\|, \|Sz - Tz, a\|, \|Bz - Bz, a\|, \|Sz - Sz, a\|, \|Sz - Bz, a\|] \ge 0$$

That is, $F[\|Sz - Bz, a\|, \|Sz - Tz, a\|, 0, 0] \ge 0$.

$\Rightarrow \quad Sz = Bz = Tz$.

Step III. By putting x = y = z in (1.2), we have

$$\|Az - Bz, a\| \leq h\left[max\begin{Bmatrix}\|Sz - Az, a\|, \|Tz - Bz, a\|, \|Sz - Tz, a\|, \\ \|Tz - Az, a\|, \|Sz - Bz, a\|\end{Bmatrix}\right]$$

$$= h[max\{\|Sz - Az, a\|, 0, 0, \|Bz - Az, a\|, 0\}]$$

That is, $\|Az - Bz, a\| \leq h[max\{\|Sz - Az, a\|, 0, 0, \|Bz - Az, a\|, 0\}]$

Thus, $Az = Bz = Sz = Tz.$

Step IV. Putting x = x_{2n}, y = z in (1.2), we have

$$\|Ax_{2n} - Bz, a\| \leq h\left[max\begin{Bmatrix}\|Sx_{2n} - Ax_{2n}, a\|, \|Tz - Bz, a\|, \|Sx_{2n} - Tz, a\|, \\ \|Tz - Ax_{2n}, a\|, \|Sx_{2n} - Bz, a\|\end{Bmatrix}\right]$$

As $n \to \infty$, we get

$$\|z - Bz, a\| \leq h\left[max\begin{Bmatrix}\|z - z, a\|, \|Bz - Bz, a\|, \|z - Bz, a\|, \\ \|Bz - z, a\|, \|z - Bz, a\|\end{Bmatrix}\right]$$

$$= h\left[max\{0, 0, \|z - Bz, a\| \}\right]$$

$$\Rightarrow Bz = z.$$

Thus $z = Az = Bz = Sz = Tz.$

Thus in this case, z is a common fixed point of A, B, S and T.

Case (iii). If either A(X) or B(X) is complete.

As $A(X) \subseteq T(X) and B(X) \subseteq S(X)$, therefore, the result follows from Case (i) and Case (ii).

From condition (1.2) we get the uniqueness.

Theorem 4.2: Let (X,‖.,.‖) be a 2-banach space such that ◊ satisfy α-property with $\alpha \geq 0$. Let A, B,S,T be self mappings of X into itself satisfy the following conditions:

1. $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ and T(X) and S(X) are closed subset of X
2. The paid (A,S) and (B,T) are weakly compatible.
3. $\|Ax - By, u\| \leq K_1[\|Sx - Ty, u\| \lozenge \|Ax - Sx, u\|] + K_2[\|Sx - Ty, u\| \lozenge \|By - Ty, u\|] + K_3\left[\|Sx - Ty, u\| \lozenge \frac{1}{2}\{\|Sx - By, u\| + \|Ax - Ty, u\|\}\right]$ for all *x*,*y* in X, where $K_1, K_2, K_3 > 0$ and $0 < K_1 + K_2 + K_3 < 1$, Then A, B, S, T have unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X.

We find a sequence $\{Y_N\}$ in X such that $Y_{2n} = A_{x_{2n}} = T_{x_{2n+1}}$ and $Y_{2n+1} = B_{x_{2n+1}} = S_{x_{2n+2}}$ for all $n = 0,1,2,\ldots$ ...... (2.1)

We claim that $\{Y_N\}$ is a Cauchy sequence.

Using (3), we get $\|Y_{2n} - Y_{2n+1}, u\| = \|A_{x_{2n}} - B_{x_{2n+1}}, u\| \leq K_1[\|Sx_{2n} - Tx_{2n+1}, u\| \lozenge \| Ax_{2n} - Sx_{2n}, u\|] + K_2[\|Sx_{2n} - Tx_{2n+1}, u\| \lozenge \|Bx_{2n+1} - Tx_{2n+1}, u\|] + K_3\left[\|Sx_{2n} - Tx_{2n+1}, u\| \lozenge \frac{1}{2}\{\|Sx_{2n} - Bx_{2n+1}, u\| + \|Ax_{2n} - Tx_{2n+1}, u\|\}\right]$

$=K_1[\|Y_{2n-1}-Y_{2n},u\| \lozenge \|Y_{2n}-Y_{2n-1},u\|]+K_2[\|Y_{2n-1}-Y_{2n},u\| \lozenge \|Y_{2n+1}-Y_{2n},u\|]+$
$K_3\left[\|Y_{2n-1}-Y_{2n},u\| \lozenge \frac{1}{2}\{\|Y_{2n-1}-Y_{2n+1},u\|+\|Y_{2n}-Y_{2n},u\|\}\right]$

Thus $\|Y_{2n}-Y_{2n+1},u\| \le K_1[\|Y_{2n-1}-Y_{2n},u\| \lozenge \|Y_{2n}-Y_{2n-1},u\|]+K_2[\|Y_{2n-1}-Y_{2n},u\| \lozenge \|Y_{2n+1}-Y_{2n},u\|]+K_3\left[\|Y_{2n-1}-Y_{2n},u\| \lozenge \frac{1}{2}\{\|Y_{2n-1}-Y_{2n+1},u\|+\|Y_{2n}-Y_{2n},u\|\}\right]$

------- (2.2)

Put $d_n=\|Y_{n-1}-Y_n,u\|$

------ (2.3)

Then, the above inequality becomes,

$$d_{2n+1} \le K_1[d_{2n} \lozenge d_{2n}]+K_2[d_{2n} \lozenge d_{2n+1}]+K_3\left[d_{2n} \lozenge \frac{1}{2}\|Y_{2n-1}-Y_{2n+1},u\|\right]$$

$$d_{2n+1} \le K_1\alpha d_{2n}+K_2\alpha\, max[d_{2n},d_{2n+1}]+K_3\alpha\, max\left[d_{2n},\frac{1}{2}\|Y_{2n-1}-Y_{2n}+Y_{2n}-Y_{2n+1},u\|\right]$$

$$\le K_1\alpha d_{2n}+K_2\alpha\, max[d_{2n},d_{2n+1}]+K_3\alpha\, max\left[d_{2n},\frac{1}{2}\{d_{2n},d_{2n+1}\}\right]$$

If $d_{2n+1}>d_{2n}$ then $d_{2n+1}<K_1\alpha d_{2n+1}+K_2\alpha d_{2n+1}+K_3\alpha d_{2n+1}$

$$d_{2n+1}<\alpha(K_1+K_2+K_3)d_{2n+1} \Rightarrow d_{2n+1}<d_{2n+1}$$

This is a contradiction. That is $d_{2n+1}<d_{2n}$. Hence, $d_{2n}<d_{2n-1}$.
Therefore, $d_n \le d_{n-1},\ for\ n=1,2,3,\dots$

$$d_n \le \alpha(K_1+K_2+K_3)d_{n-1} \Rightarrow d_n \le Kd_{n-1}, where\ K=\alpha(K_1+K_2+K_3)<1$$

By iteration n times, we get $d_n \le Kd_{n-1} \le K^2 d_{n-2} \le \cdots \le K^n d_0$.

Taking the limit as $n\to\infty$, we get $\lim_{n\to\infty}\|Y_{n-1}-Y_n,u\|=0$ ------- (2.4)

Let $m>n$ such that $m=2n+1$

We claim that $\{Y_n\}$ is a Cauchy sequence.

Suppose that n is the least integer for which $\|Y_n-Y_m,u\| \ge \varepsilon$

But $\|Y_{n-1}-Y_m,u\|<\varepsilon$ ...... (2.5)

Now $\varepsilon<\|Y_n-Y_m,u\| \le \|Y_n-Y_m,u\|+\|Y_n-Y_{n-1},u\|+\|Y_{n-1}-Y_m,u\|$ ...... (2.6)

Now taking the term $\|Y_n-Y_m,Y_{n-1}\|$

$$\|Y_n-Y_m,Y_{n-1}\|=\|Ax_n-Bx_m,Y_{n-1}\| \le K_1[\|Sx_n-Tx_m,Y_{n-1}\| \lozenge \|Ax_n-Sx_n,Y_{n-1}\|]$$
$$+\ K_2[\|Sx_n-Tx_m,Y_{n-1}\| \lozenge \|Bx_m-Tx_m,Y_{n-1}\|]$$
$$+K_3\left[\|Sx_n-Tx_m,Y_{n-1}\| \lozenge \frac{1}{2}\{\|Sx_n-Bx_m,Y_{n-1}\|+\|Ax_n-Tx_m,Y_n\|\}\right]$$
$$=K_1[\|Y_{n-1}-Y_{m-1},Y_{n-1}\| \lozenge \|Y_n-Y_{m-1},Y_{n-1}\|]+K_2[\|Y_{n-1}-Y_{m-1},Y_{n-1}\| \lozenge \|Y_m-Y_{m-1},Y_{n-1}\|]$$
$$+K_3\left[\|Y_{n-1}-Y_{m-1},Y_{n-1}\| \lozenge \frac{1}{2}\{\|Y_{n-1}-Y_m,Y_{n-1}\|+\|Y_n-Y_m,Y_{n-1}\|\}\right]$$

$\Rightarrow \|Y_n-Y_m,Y_{n-1}\| \le K_2\alpha\|Y_{n-1}-Y_m,Y_{m-1}\|+K_3\alpha^2\|Y_{n-1}-Y_m,Y_n\|$

Using (2.1) and (2.2) and taking the limit as $n\to\infty$, we get $\|Y_n-Y_m,Y_{n-1}\|=0$ ------(2.7)

Substituting (2.4), (2.5) and (2.7) in (2.6) we get $\varepsilon < \varepsilon$, which is a contradiction.

Hence, $\{Y_n\}$ is a Cauchy Sequence.

Since X is a complete 2-metric space, $\lim_{n\to\infty} Y_n = Y \in X$. ...... (2.8)

Hence, $\lim_{n\to\infty} Y_n = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = \lim_{n\to\infty} Tx_{2n+1} = Y$

...... (2.9)

Since, T(x) is a closed subset of X then there exist $v \in X$ such that $Tv = Y$.

If $Bv \neq y$ then by using (3),

$$\|Ax_{2n} - Bv, u\| \le K_1[\|Sx_{2n} - Tv, u\| \lozenge \|Ax_{2n} - Sx_{2n}, u\|]$$
$$+K_2[\|Sx_{2n} - Tv, u\| \lozenge \|Bv - Tv, u\|] +$$
$$K_3\left[\|Sx_{2n} - Tv, u\| \lozenge \frac{1}{2}\{\|Sx_{2n} - Bv, u\| + \|Ax_{2n} - Tv, u\|\}\right]$$

Taking the limit as $n \to \infty$ both sides, we get

$$\|y - Bv, u\| \le K_1[\|y - y, u\| \lozenge \|y - y, u\|] + K_2[\|y - y, u\| \lozenge \|Bv - y, u\|]$$
$$+ K_3\left[\|y - y, u\| \lozenge \frac{1}{2}\{\|y - Bv, u\| + \|y - y, u\|\}\right]$$

$$= K_1[0 \lozenge 0] + K_2[0 \lozenge \|Bv - y, u\|] + K_3\left[0 \lozenge \frac{1}{2}\{\|y - Bv, u\| + 0\}\right]$$

$$\le K_1\alpha[\max(0,0)] + K_2\alpha[\max(0, \|Bv - y, u\|] + K_3\alpha[\max(0, \|y - Bv, u\|]$$

$$= K_2\alpha\|Bv - y, u\| + K_3\alpha\|y - Bv, u\|$$

$$= (K_2 + K_3)\alpha\|y - Bv, u\|$$

Thus, $\|y - Bv, u\| \le (K_2 + K_3)\alpha\|y - Bv, u\|$

$$\Rightarrow \|y - Bv, u\| \le \|y - Bv, u\|$$

This is a contradiction. Hence, $Bv = y = Tv$.

Since B and T are weakly compatible,

$BTv = TBv \Rightarrow By = Ty$. ...... (2.10)

Now if $y \neq By$ then by using (3),

$$\|Ax_{2n} - By, u\| \le K_1[\|Sx_{2n} - Ty, u\| \lozenge \|Ax_{2n} - Sx_{2n}, u\|]$$
$$+K_2[\|Sx_{2n} - Ty, u\| \lozenge \|By - Ty, u\|] +$$
$$K_3\left[\|Sx_{2n} - Ty, u\| \lozenge \frac{1}{2}\{\|Sx_{2n} - By, u\| + \|Ax_{2n} - Ty, u\|\}\right]$$

Taking the limit as $n \to \infty$ both sides, we get

$$\|y - By, u\| \leq K_1[\|y - Ty, u\| \Diamond \|y - y, u\|] + K_2[\|y - Ty, u\| \Diamond \|By - Ty, u\|] + K_3\left[\|y - Ty, u\| \Diamond \frac{1}{2}\{\|y - By, u\| + \|y - Ty, u\|\}\right]$$

$$= K_1[\|y - Ty, u\| \Diamond 0] + K_2[\|y - Ty, u\| \Diamond 0] + K_3\left[\|y - Ty, u\| \Diamond \frac{1}{2}\{\|y - By, u\| + \|y - Ty, u\|\}\right]$$

$$= K_1[\|y - Ty, u\| \Diamond 0] + K_2[\|y - Ty, u\| \Diamond 0] + K_3\left[\|y - Ty, u\| \Diamond \frac{1}{2}\{\|y - Ty, u\| + \|y - Ty, u\|\}\right]$$

$$= K_1[\|y - Ty, u\| \Diamond 0] + K_2[\|y - Ty, u\| \Diamond 0] + K_3[\|y - Ty, u\| \Diamond \|y - Ty, u\|]$$

$$\leq K_1\alpha[\max(\|y - Ty, u\|, 0)] + K_2\alpha[\max(\|y - Ty, u\|, 0] + K_3\alpha[\max(\|y - Ty, u\|, \|y - Ty, u\|)]$$

$$= K_1\alpha\|y - Ty, u\| + K_2\alpha\|y - Ty, u\| + K_3\alpha\|y - Ty, u\|$$

$$= (K_1 + K_2 + K_3)\alpha\|y - Ty, u\|$$

Thus, $\|y - Ty, u\| \leq (K_1 + K_2 + K_3)\alpha\|y - Ty, u\|$

That is, $\|y - By, u\| \leq (K_1 + K_2 + K_3)\alpha\|y - By, u\|$

$$\Rightarrow \|y - By, u\| \leq \|y - By, u\|$$

This is a contradiction. Hence, $By = y = Ty$.

Since, $B(X) \subseteq S(X)$ there exists $w \in X$ such that $Sw = y$. ...... (2.11)

If Aw$\neq y$ then by using (3),

$$\|Aw - By, u\| \leq K_1[\|Sw - Ty, u\| \Diamond \|Aw - Sw, u\|] + K_2[\|Sw - Ty, u\| \Diamond \|By - Ty, u\|] + K_3\left[\|Sw - Ty, u\| \Diamond \frac{1}{2}\{\|Sw - By, u\| + \|Aw - Ty, u\|\}\right]$$

$$\leq K_1[\|y - y, u\| \Diamond \|Aw - y, u\|] + K_2[\|y - y, u\| \Diamond \|y - y, u\|] + K_3\left[\|y - y, u\| \Diamond \frac{1}{2}\{\|y - y, u\| + \|Aw - y, u\|\}\right]$$

$$\|Aw - y, u\| \leq K_1[\|y - y, u\| \Diamond \|Aw - y, u\|] + K_2[0 \Diamond 0] + K_3\left[0 \Diamond \frac{1}{2}\{0 + \|Aw - y, u\|\}\right]$$

$$\leq K_1\alpha[\max(0, \|Aw - y, u\|)] + K_2\alpha[\max(0,0)] + K_3\alpha[\max(0, \|Aw - y, u\|]$$

$$= K_1\alpha\|Aw - y, u\| + K_3\alpha\|Aw - y, u\|$$

$$= K_1\alpha\|Aw - y, u\| + K_3\alpha\|Aw - y, u\|$$

$$= (K_1 + K_3)\alpha\|Aw - y, u\|$$

Thus, $\|Aw - y, u\| \leq (K_1 + K_3)\alpha\|Aw - y, u\|$

$$\Rightarrow \|Aw - y, u\| \leq \|Aw - y, u\|$$

This is a contradiction. Hence, $Aw = y \Rightarrow Sw = y = Aw$.

Since S and A are weakly compatible,

$$ASw = SAw \Rightarrow Sy = Ay. \qquad \text{...... (2.12)}$$

If $Ay \neq y$ then by using (3),

$$\|Ay - y, u\| = \|Ay - By, u\| \le K_1[\|Sy - Ty, u\| \lozenge \|Ay - Sy, u\|]$$
$$+K_2[\|Sy - Ty, u\| \lozenge \|By - Ty, u\|] +$$
$$K_3\left[\|Sy - Ty, u\| \lozenge \frac{1}{2}\{\|Sy - By, u\| + \|Ay - Ty, u\|\}\right]$$

$$= K_1[\|Ay - y, u\| \lozenge \|Ay - Ay, u\|]$$
$$+K_2[\|Ay - y, u\| \lozenge \|y - y, u\|] +$$
$$K_3\left[\|Ay - y, u\| \lozenge \frac{1}{2}\{\|Ay - y, u\| + \|Ay - y, u\|\}\right]$$

$$\|Ay - y, u\| \le K_1[\|Ay - y, u\| \lozenge 0] + K_2[\|Ay - y, u\| \lozenge 0]$$
$$+ K_3[\|Ay - y, u\| \lozenge \|Ay - y, u\|]$$

$$\le K_1\alpha[\max(0, \|Ay - y, u\|)] + K_2\alpha[\max(\|Ay - y, u\|, 0)] + K_3\alpha[\max(0, \|Ay - y, u\|]$$

$$= K_1\alpha\|Ay - y, u\| + K_2\alpha\|Ay - y, u\| + K_3\alpha\|Ay - y, u\|$$

$$= (K_1 + K_2 + K_3)\alpha\|Ay - y, u\|$$

Thus, $\|Ay - y, u\| \le (K_1 + K_2 + K_3)\alpha\|Ay - y, u\|$

$$\Rightarrow \|Ay - y, u\| \le \|Ay - y, u\|$$

This is a contradiction. Hence, $Ay = y \Rightarrow Sy = y = Ay$. ...... (2.13)

Using all these, we get

$$Ay = By = Sy = Ty = y.$$

That is, y is a common fixed point for A, B, S, T.

Uniqueness:

Let A, B, S, T have another fixed point x (say) then $\|x - y, u\| = \|Ax - Bx, u\|$

$$\le K_1[\|Sx - Ty, u\| \lozenge \|Ax - Sx, u\|] + K_2[\|Sx - Ty, u\| \lozenge \|By - Ty, u\|]$$
$$+K_3\left[\|Sx - Ty, u\| \lozenge \frac{1}{2}\{\|Sx - By, u\| + \|Ax - Ty, u\|\}\right]$$
$$\le K_1[\|x - y, u\| \lozenge \|x - x, u\|] + K_2[\|x - y, u\| \lozenge \|y - y, u\|]$$
$$+K_3\left[\|x - y, u\| \lozenge \frac{1}{2}\{\|x - y, u\| + \|x - y, u\|\}\right]$$

$$= K_1[\|x - y, u\| \lozenge 0] + K_2[\|x - y, u\| \lozenge 0] + K_3[\|x - y, u\| \lozenge \|x - y, u\|]$$

$$\leq K_1\alpha[\max(\|x-y,u\|,0)] + K_2\alpha[\max(\|x-y,u\|,0)] + K_3\alpha[\max(\|x-y,u\|,\|x-y,u\|]$$

$$= K_1\alpha\|x-y,u\| + K_2\alpha\|x-y,u\| + K_3\alpha\|x-y,u\|$$

$$= (K_1+K_2+K_3)\alpha\|x-y,u\|$$

$$\Rightarrow \|x-y,u\| \leq \|x-y,u\|$$

This is again a contradiction. These contradictions imply that A, B, S and T have a unique common fixed point.

Corollary 4.3

Let (X,$\|.,.\|$) be a 2-banach space. Let A, B, S and T be self mappings of X into itself satisfy the following conditions:

1. $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ and T(X) and S(X) are closed subset of X
2. The paid (A,S) and (B,T) are weakly compatible.
3. $\|Ax-By,z\| \leq K_1[\|Sx-Ty,z\| + \|Ax-Sx,z\|] + K_2[\|Sx-Ty,z\| + \|By-Ty,z\|] + K_3\left[\|Sx-Ty,z\| + \frac{\|Sx-By,u\|+\|Ax-Ty,u\|}{2}\right]$ For all x,y in X, where $K_1, K_2, K_3 > 0$ and $0 < K_1+K_2+K_3 < \frac{1}{2}$. Then A, B, S, T have unique common fixed point in X.

Proof: Define $a \lozenge b = a + b$ for each $a, b \in R^+$. Then for $\alpha \geq 2$, we have $a \lozenge b \leq \alpha\, max\{a, b\}$. Putting $\alpha = 2$, we get $0 < \alpha(K_1+K_2+K_3) < 1$, and all conditions of above theorem hold. Therefore $A, B, S\ and\ T$ have a unique common fixed point in X.

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